On Vertex Offsets of Polyhedral Surfaces

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Abstract

Planar-faced mesh surfaces, also known as polyhedral surfaces, that possess vertex-offsets are useful in architectural design for constructing supporting structures, and also of interest in discrete differential geometry. We consider the existence and computation of vertex-offset meshes of general polyhedral surfaces and, specifically, study how the existence of vertex offsets is dictated by the face shape, mesh surface geometry and mesh surface topology. This extends the study in [Pottmann et al. 2007; Pottmann and Wallner 2008] on vertex-offset meshes of simply-connected circular quad meshes.

Keywords: vertex offset, quasi-circular mesh, mesh parallelism

1 Introduction

Polyhedral surfaces are meshes with planar faces. It has recently been shown [Liu et al. 2006] that polyhedral surfaces, especially those beyond triangle meshes, are useful for modeling glass/metal panels in architectural design. A vertex-offset mesh of a given mesh $\mathcal{M}_0$ is one that has the same mesh connectivity of $\mathcal{M}_0$ and all of its vertices have a same constant distance to their corresponding vertices in $\mathcal{M}_0$. The vertex-offset mesh, along with other variants of offset meshes, are useful for building supporting structures of a building surface modeled as a polyhedral surface [Pottmann et al. 2007].

A circular quad mesh is a mesh with planar quad faces each of which has a circum-circle. Circular meshes were first introduced by Martin et al. [1986] as quad meshes with planar faces which discretize the principal curvature lines of an underlying smooth surface and possess a circum-circle for each quad face. Recently circular meshes have been well studied from the discrete differential geometry point of view [Bobenko and Suris 2005]. The focal geometry of circular meshes, including discrete normals, offsets and focal surfaces, has recently been studied in [Pottmann and Wallner 2008].

A simply connected quad mesh surface possesses a vertex-offset mesh if and only if it is a circular quad mesh [Pottmann et al. 2007]; in fact, a simply connected circular quad mesh has a two-parameter family of parallel spherical meshes [Pottmann and Wallner 2008]. However, the existence of the vertex-offset mesh of a general polyhedral surface is a more complex problem. For example, a quad mesh with more than one closed boundaries or a closed quad mesh with nonzero genus may not have vertex-offset meshes, except for the trivial case of a translation of the mesh. Two examples are shown in Figure 1.

We study in this paper how the existence of vertex-offset meshes of a polyhedral surface is governed by the face shape, mesh surface geometry and mesh surface topology. We also present numerical techniques for computing the parallel spherical meshes of a vertex-offset mesh.

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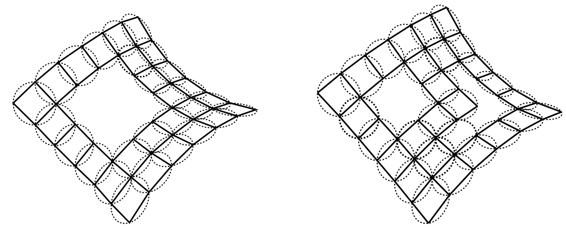

Figure 1: *Both quad meshes are circular meshes. The one on the left has a unique parallel spherical mesh, and the one one the right has no parallel spherical mesh and therefore no vertex offset.*

2 Quasi-circular polygons and meshes

The vertex offset can be characterized using the concept of parallel meshes [Pottmann et al. 2007] – a mesh $\mathcal{M}_0$ possesses a vertex-offset mesh if and only if it has a parallel spherical mesh $\mathcal{M}_s$, which is a mesh that has all of its vertices lie on the unit sphere S^2. Such a mesh $\mathcal{M}_s$ is called a parallel spherical embedding mesh of $\mathcal{M}_0$. Based on this result, our study will be centered around the existence of the spherical embedding meshes of a given mesh. We first introduce necessary notions and definitions.

A mesh $\mathcal{M}$ is a *planar polygonal mesh*, or called *a polyhedral surface*, if every face of $\mathcal{M}$ spans a plane. Since we always require face planarity, all meshes discussed in the paper will be assumed to have planar faces. A *proper mesh* is one such that it does not have degenerate edges, that is, the two endpoints of any edge are not identical.

Two polygons are *parallel* if their vertices can be put in one-to-one correspondence such that the corresponding edges are parallel. A *circular polygon* is a planar polygon inscribed in a circle. A *circular mesh* is a mesh all of whose faces are circular polygons. To study the vertex-offset mesh of a general mesh in the framework of parallel meshes, we need to go beyond circular meshes.

Definition 1 *A planar polygon $\mathcal{P}$ is called a* quasi-circular polygon *if it has a parallel non-degenerate circular polygon $\mathcal{P}_c$ inscribed in a unit circle S^1. Here the non-degeneracy of $\mathcal{P}_c$ means that if the two endpoints u_c and v_c of any edge e_c of $\mathcal{P}_c$ are identical, then the direction of e_c's corresponding edge e in $\mathcal{P}$ is tangent to the circle S^1 at u_c $(= v_c)$. We call $\mathcal{P}_c$ a* circular embedding polygon *of $\mathcal{P}$.*

Note that any triangle is circular, since it has a unique circum-circle. A planar quad is circular if the sums of two opposite internal angles are equal. A quasi-circular quad mesh $\mathcal{P}$ has infinitely many circular embedding polygons, including itself; therefore, a quasi-circular quad is also circular. However, a quasi-circular polygon with more than four sides, in general, is not circular.

Definition 2 *A mesh is called a* quasi-circular mesh *if all of its faces are quasi-circular polygons.*

The notion of quasi-circular meshes is the equivalent to that of circular meshes for quad meshes, but different from and more general than the latter for mesh faces that contain faces with more than four sides.

Two polygonal meshes are defined to be *parallel* if their vertices are in one-to-one correspondence with isomorphic edge connectivity and their corresponding edges are parallel. Clearly, the corresponding faces of two parallel meshes are parallel.

Definition 3 *A mesh is a* spherical mesh *if all of its vertices lie on the unit sphere S^2.*

The existence of the vertex-offset meshes of a given mesh can be characterized by the existence of its parallel spherical meshes, as observed in [Pottmann et al. 2007]. This is summarized as follows.

Proposition 1 *A mesh $\mathcal{M}$ has a vertex-offset mesh if and only if it has a non-degenerate parallel spherical mesh $\mathcal{M}_s$. Here a "non-degenerate" $\mathcal{M}_s$ means that if the two endpoints u_s and v_s of any edge e_s of $\mathcal{M}_s$ are identical, then the direction of e_s's corresponding edge e in $\mathcal{M}$ is parallel to the tangent plane of S^2 at u_s ($= v_s$).*

We also call $\mathcal{M}_s$ the parallel spherical embedding of $\mathcal{M}$. With $\mathcal{M}_s$, a vertex-offset mesh $\mathcal{M}_o$ of $\mathcal{M}$ can be expressed as $\mathcal{M}_o := \mathcal{M} + d\mathcal{M}_s$, for some constant d and "+" is vector addition applied to the coordinates of the corresponding vertices. By insisting on the non-degeneracy of $\mathcal{M}_s$, we have excluded the trivial case where $\mathcal{M}_o$ is a translational copy of $\mathcal{M}$, which would be caused by $\mathcal{M}_s$ collapsing into a single point on S^2.

It follows from Proposition 1 that a mesh possessing a vertex-offset mesh is a quasi-circular mesh. This conditions turns out to be also sufficient for a simply connected circular quad mesh, as pointed above, since a simply connected circular quad mesh always has a two-parameter family of parallel spherical meshes [Pottmann and Wallner 2008].

We now consider the 2D reflections induced by the sides a quasi-circular polygon $\mathcal{P} : \mathrm{u}_0\mathrm{u}_1 \cdots \mathrm{u}_n$, with $\mathrm{u}_0 = \mathrm{u}_n$. Similar to the procedure described in Remark 11 in [Pottmann and Wallner 2008], the construction of a circular embedding polygon $\mathcal{P}' : \mathrm{u}'_0\mathrm{u}'_1 \cdots \mathrm{u}'_n$ of $\mathcal{P}$ can be obtained via a series of reflections. The procedure is as follows:

- Pick a point u'_0 on the unit circle, set $j = 0$;
- Repeat the following procedure until $j = n$: Select the line passing through the origin and orthogonal to $\overrightarrow{\mathrm{u}_{j+1}\mathrm{u}_j}$ as the reflection line. Reflect u'_j in this line to obtain u'_{j+1}. (This reflection is denoted as T_j which is a 2×2 orthogonal matrix.) Set $j := j + 1$.

If $\mathcal{P}$ is a quasi-circular polygon, then there needs to be $\mathrm{u}'_0 = \mathrm{u}'_n$ in order for $\mathcal{P}'$ to exist. Define $T_{\texttt{ref}} := T_{n-1} \circ T_{n-2} \circ \cdots \circ T_1 \circ T_0$. Then $\mathrm{u}'_0 = \mathrm{u}'_n$ implies $\mathrm{u}'_0 = T_{\texttt{ref}}\mathrm{u}'_0$, or equivalently, $T_{\texttt{ref}}$ has an eigenvalue equal to 1 and u'_0 is the associated eigenvector. Since each reflection T_j is a 2×2 orthogonal matrix with $\det(T_j) = -1$, $T_{\texttt{ref}}$ is orthogonal with $\det(T_{\texttt{ref}}) = -1$ when n is odd and $\det(T_{\texttt{ref}}) = 1$ when n is even.

When n is odd, the two eigenvalues of $T_{\texttt{ref}}$ are 1 and -1. That is to say, there is a unique invariant vector u'_0 of $T_{\texttt{ref}}$ in this case, up to scaling. Hence, there are exactly two circular embedding polygons of $\mathcal{P}$, given by a unit invariant vector u'_0 of $T_{\texttt{ref}}$ and $-\mathrm{u}'_0$. We state this as a proposition.

Proposition 2 *Any odd-sided planar polygon is a quasi-circular polygon with two circular embedding polygons, which are reflections of each other about the center of the circle.*

When n is even, it is easy to see that $T_{\texttt{ref}} = I_{2\times 2}$ (the identity matrix) if and only if $\mathcal{P}$ is quasi-circular. In this case, any point u'_0 on the unit circle S^1 gives an eigenvalue vector of $T_{\texttt{ref}}$; therefore, there is a one-parameter family of circular embedding polygons of $\mathcal{P}$. When n is even and $\mathcal{P}$ is not quasi-circular, the eigenvalues of $T_{\texttt{ref}}$ are complex conjugate or -1 and -1, representing a 2D rotation of angle not equal to a multiple of 2π.

A quasi-circular even-sided convex polygon is characterized by the following theorem (we do not include its proof here due to space limit).

Theorem 1 *An even-sided planar polygon $P = \mathrm{u}_1\mathrm{u}_2 \ldots \mathrm{u}_{2n}$ is a quasi-circular if and only if its internal angles $\theta_1, \theta_2, \ldots, \theta_{2n}$ satisfy $\theta_1 + \theta_3 + \ldots + \theta_{2n-1} = \theta_2 + \theta_4 + \ldots + \theta_{2n}$.*

Proposition 3 *An even-sided quasi-circular polygon has a one-parameter family of circular embedding polygons.*

Consider a quasi-circular (planar) polygon in 3D. The reflections along its successive sides define a series of reflections in 3D following a similar procedure: we reflect the vertex u'_j with respect to the plane with normal vector $\overrightarrow{\mathrm{u}_{i+1}\mathrm{u}_i}$ which passes through the origin to obtain u'_{j+1}. All possible combinations of the eigenvalues of $T_{\texttt{ref}}$ are listed in the following table.

A quasi-circular polygon in 3D	$\det(T_{\texttt{ref}}) = 1$ (n is even)	$\det(T_{\texttt{ref}}) = -1$ (n is odd)
Eigenvalues	$\{1, 1, 1\}$	$\{1, 1, -1\}$

It follows that an even-sided quasi-circular polygon $\mathcal{P}$ has a two-parameter family of parallel embedding polygons on the unit sphere S^2; in this case any point on S^2 can be used as a starting point u'_0 to construct an embedding $\mathcal{P}'$ of $\mathcal{P}$ on S^2. For an odd-sided planar polygon, which is always quasi-circular (cf. Proposition 2), there is a one-parameter family of parallel embedding polygons on S^2. In this case, the initial point u'_0 of $\mathcal{P}'$ is confined to be on a great arc of S^2. Hence, we conclude that if a mesh $\mathcal{P}$ contains more than two odd-sided faces in general orientations, then $\mathcal{P}$, in general, does not have a parallel spherical embedding and therefore has no vertex-offset meshes, since three great circles on S^2, in general, do not have a common point.

This suggests that meshes with odd-sided faces, such as triangle meshes and pentagonal meshes, are not amendable to vertex-offset computation. Hence, from now on we will only consider quasi-circular meshes with even-sided face, which will be referred to as a *quasi-circular even-sided polygonal mesh*, or *QCEP mesh* for short.

3 Spherical loops and loop homotopy

A quasi-circular mesh is characterized by the local condition that each face is quasi-circular. Except in the case of a simply connected quasi-circular mesh, the existence of the vertex-offset of a general quasi-circular mesh is determined globally by the shape and topology of the mesh. Here, the topology refers to the number of closed boundaries for an open mesh surface and the genus for a closed mesh surface. To study this topological aspect, we need to consider loops on the mesh.

A *loop* $\mathcal{L}$ on a mesh $\mathcal{P}$ is a closed path consisting of a sequence of incident edges of $\mathcal{P}$; therefore, it is, in fact, a closed polygon in 3D, which is not necessarily planar. Parallel loops are defined in the same way as for parallel planar polygons. All loops on a mesh surface can be classified into equivalence classes via homotopy – two loops are homotopic if they can deform continuously into each other on the surface. Evidently, a loop $\mathcal{L}_1$ on a quasi-circular EP mesh $\mathcal{P}$ can be deformed into another loop $\mathcal{L}_2$ homotopic to $\mathcal{L}_1$ via a sequence of face addition or face removal operations, one face at a time.

Definition 4 *A loop $\mathcal{L}$ is* quasi-spherical *if it has a non-degenerate parallel loop $\mathcal{L}'$ whose vertices are on the unit sphere S^2. Here a "non-degenerate" parallel loop $\mathcal{L}'$ means that if the two endpoints u_s and v_s of any edge e_s of the parallel loop $\mathcal{L}'$ are identical, then the direction of e_s's corresponding edge e in $\mathcal{L}$ is parallel to the* *tangent plane of S^2 at u_s $(= v_s)$. The loop $\mathcal{L}'$ is called a* parallel spherical embedding loop *of $\mathcal{L}$.*

The next theorem implies that if a loop on a quasi-circular mesh is quasi-spherical, then all loops homotopic to it are quasi-spherical. We skip the proof due to space limit.

Theorem 2 *Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are homotopic on a quasi-circular even-sided (QCEP) mesh $\mathcal{P}$. Then $\mathcal{L}_1$ is quasi-spherical if and only if $\mathcal{L}_2$ is quasi-spherical. Furthermore, when $\mathcal{L}_1$ is quasi-spherical, all the parallel spherical embedding loops of $\mathcal{L}_1$ are consistent and in one-to-one correspondence with those of $\mathcal{L}_2$.*

Clearly, any loop on a mesh which has a parallel spherical embedding is quasi-spherical. Therefore, if there is a non-quasi-spherical loop on a quasi-circular mesh, then the mesh does not have a vertex-offset mesh.

A loop induces a sequence of 3D reflections along it sides. Whether or not a loop is quasi-spherical can be analyzed by considering the eigenvalues of the composition of the reflections, again denoted as $T_{\mathtt{ref}}$. All possible combinations of the eigenvalues of $T_{\mathtt{ref}}$ for an n-sided loop in 3D are listed in the following table.

A loop in 3D	$\det(T_{\mathtt{ref}}) = 1$ (n is even)	$\det(T_{\mathtt{ref}}) = -1$ (n is odd)
Eigenvalues	$\{1, 1, 1\}$ $\{1, -1, -1\}$ $\{1, a + b\mathbf{i}, a - b\mathbf{i}\}$	$\{1, 1, -1\}$ $\{-1, -1, -1\}$ $\{-1, a + b\mathbf{i}, a - b\mathbf{i}\}$

Here $a^2 + b^2 = 1, a, b \neq 0$ and $a, b \in \mathbb{R}$.

Hence, an even-sided loop is always quasi-spherical, since $T_{\mathtt{ref}}$ has at least one eigenvalue equal to 1. In the special case where the eigenvalues are $\{1, 1, 1\}$, the loop has a two-parameter family of parallel spherical embedding loops. An odd-sided loop may not be quasi-spherical, since it may not have an eigenvalue equal to 1. Only in the special case where the eigenvalues are $\{1, 1, -1\}$, the loop has a one-parameter family of parallel spherical embedding loops; this happens, for instance, for an odd-sided planar polygon.

4 Existence of vertex-offset meshes

Open quasi-circular meshes For a simply connected open QCEP mesh, all loops on it are simply connected, even-sided, and homotopic to each other. In particular, every loop is homotopic to every face of $\mathcal{M}$, which is even-sided and quasi-circular with a two-parameter family of spherical embedding polygons. Hence, any simply connected QCEP mesh has a two-parameter family of parallel spherical embedding meshes.

Now consider an open QCEP mesh $\mathcal{M}$ with one hole, which has the topology of a truncated cylinder, as shown for example in Figure 1 (left). Clearly, a simply connected loop $\mathcal{L}_0$ on $\mathcal{M}$ is even-sided, but a non-simply connected loop $\mathcal{L}_1$ around the hole (with winding number equal to 1) may be even-sided or odd-sided. If $\mathcal{L}_1$ is even-sided, then $\mathcal{L}_1$ is quasi-spherical and there are in general two parallel spherical meshes of $\mathcal{L}_1$ that are reflections of each other about the center of the sphere, resulting from a unit invariant eigenvector u_0' of $T_{\mathtt{ref}}$ and $-\mathrm{u}_0'$; two such spherical meshes are said to be *diametrically opposite*, and they lead to the unique family of vertex offset meshes $\mathcal{M}_o := \mathcal{M} + d\mathcal{M}_s$ with the parameter d. It follows that the mesh $\mathcal{P}$ in general has a unique vertex-offset mesh, since any other loop around the hole as an element of the fundamental group of $\mathcal{M}$ can be generated from the generators $\mathcal{L}_0$ and $\mathcal{L}_1$. In the special case where $T_{\mathtt{ref}} = I_{3\times 3}$ for $\mathcal{L}_1$, $\mathcal{M}$ has a two-parameter family of parallel spherical meshes. When $\mathcal{L}_1$ is odd-sided, $\mathcal{M}$ in general does not have a vertex-offset mesh.

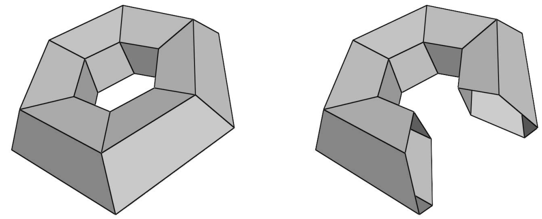

Figure 2: *A torus-like QCEP mesh without a vertex-offset mesh (left). The same surface but with one row removed for better illustration.*

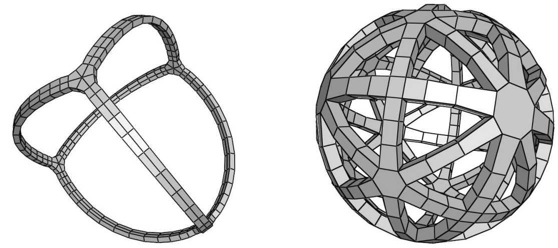

Figure 3: *Two QCEP meshes without vertex-offset meshes. The one on the left has non-homotopic quasi-spherical loops whose parallel spherical embedding loops are inconsistent with each other. The one on the right has odd-sided loops that are not quasi-spherical. The original models are from the repository of TOPMOD.*

When an open QCEP mesh $\mathcal{M}$ has more than one hole, there are at least three closed boundary loops that are not homotopic to each other. Then $\mathcal{M}$ in general does not have any vertex-offset mesh, even if each loop is quasi-spherical, since the spherical embedding loops of these loops may not have a common vertex.

Closed quasi-circular meshes The topology of a closed QCEP mesh is indicated by its genus, i.e., the number of its handles. First, if a QCEP mesh $\mathcal{M}$ is a topological sphere, then it is simply connected. Consequently, we can show that it has a two-parameter family of vertex-offset meshes.

For a QCEP mesh $\mathcal{M}$ of genus 1 or higher, there are at least three loops that are mutually non-homotopic. It follows that $\mathcal{M}$ in general does not has a vertex-offset mesh, either because one of these loops is not quasi-spherical or because the parallel spherical embedding loops of these non-homotopic loops are inconsistent even if they are all quasi-spherical. Figure 2 shows a QCEP mesh that does not have a vertex offset, because it contains a non-planar odd-sided loop that is non-quasi-spherical. Figure 3 shows two more QCEP meshes with high genus that have no vertex-offset meshes.

For a closed QCEP mesh $\mathcal{M}$ of genus $g > 0$, if all the generator loops of the fundamental group of $\mathcal{M}$ are even-sided, quasi-spherical and have their composed reflections $T_{\mathtt{ref}} = I_{3\times 3}$, then $\mathcal{M}$ has a two-parameter family of spherical meshes, and hence a two-parameter family of vertex offsets.

Finally, the above observations also apply to a QCEP mesh surface with open boundaries and handles.

5 Numerical computation of parallel spherical meshes

In the preceding sections we considered the existence of parallel spherical meshes of a quasi-circular mesh via an analysis of mesh topology. Checking whether each loop is quasi-circular is not an

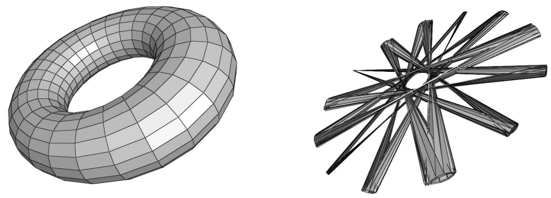

Figure 4: *A torus-like circular quadrilateral mesh (left) with its unique parallel spherical mesh (right).*

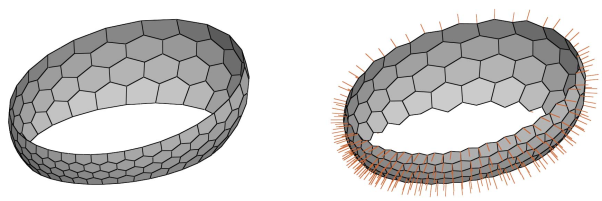

Figure 5: *Left: A hexagonal mesh before optimization; right: A quasi-circular hexagonal mesh after optimization with its discrete normal.*

efficient way to determine the existence of the parallel spherical mesh. In this section, we utilize mesh parallelism and reflectional properties to compute the basis of the parallel spherical mesh.

Denote a given quasi-circular polygonal mesh as $\mathcal{M}$ and its parallel spherical mesh be denoted as $\mathcal{M}_s$. Denote the edges of $\mathcal{M}$ and $\mathcal{M}_s$ as e_k and e'_k, $k = 1, \ldots, N_e$. The two end vertices of e_k and e'_k are denoted as $\mathrm{v}_{k,1}, \mathrm{v}_{k,2}$ and $\mathrm{v}'_{k,1}, \mathrm{v}'_{k,2}$, respectively. Since the corresponding edges of $\mathcal{M}$ and $\mathcal{M}_s$ are parallel, we have

$$(\mathrm{v}_{k,2} - \mathrm{v}_{k,1}) \times (\mathrm{v}'_{k,2} - \mathrm{v}'_{k,1}) = \mathbf{0}, \; k = 1, \ldots, N_e, \tag{1}$$

where "$\times$" stands for vector vector cross-product in 3D. Since $\mathrm{v}'_{k,1}$ and $\mathrm{v}'_{k,2}$ on S^2 are related by a reflection in a plane with the normal vector $\mathrm{v}_{k,2} - \mathrm{v}_{k,1}$, we have

$$\frac{\mathrm{v}'_{k,1} + \mathrm{v}'_{k,2}}{2} \cdot (\mathrm{v}_{k,2} - \mathrm{v}_{k,1}) = 0, \; k = 1, \ldots, N_e, \tag{2}$$

where "$\cdot$" stands for inner-product. Note that the last equation also ensures the non-degeneracy of $\mathcal{M}_s$ with respect to $\mathcal{M}$ when $\mathrm{v}'_{k,1} = \mathrm{v}'_{k,2}$ (cf. Proposition 1). Thus we obtain a homogenous sparse linear system with size $4N_e \times 3N_v$, where N_v is the number of vertices and v' are unknowns. The space of the vertices of $\mathcal{M}_s$ is the null space of the sparse system of the linear equations, which can be solved by SVD efficiently. Figure 4 shows an example.

Meshes produced from architects are often not circular, and not even planar, in general. If one wishes to compute a vertex offset mesh of such a mesh, one may use the numerical optimization technique presented in [Liu et al. 2006; Pottmann et al. 2007]. For meshes with even-sided faces, face quasi-circularity can be attained with optimization based on the condition of equal angle sums as stated in Theorem 1. (For reasons stated previously, we will not consider here circular meshes with odd-sided faces.) For example, using the optimization technique similar to that presented in [Liu et al. 2006; Pottmann et al. 2007] via mesh vertex perturbation, one can turn a given hexagonal mesh into a quasi-circular planar hexagonal mesh by enforcing $\sum_{i=1}^{6} \theta_i = 4\pi$ (face planarity) and $\theta_1 + \theta_3 + \theta_5 = \theta_2 + \theta_4 + \theta_6$ (quasi-circularity) for each face. An example is shown in Figure 5.

Based on Theorem 1 and our numerical optimization technique, we can also handle a quasi-circular mesh with mixed types of faces. For example, Figure 6 shows a simply connected QCEP mesh with three types of quasi-circular faces. This mesh has a two-parameter family of vertex-offset meshes.

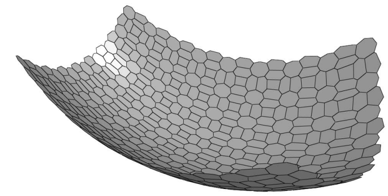

Figure 6: *A QCEP polygonal mesh with three types of faces – quadrilateral, hexagonal and octagonal faces. This mesh also possesses a face-offset mesh, since each of its interior vertices has valence 3.*

6 Conclusion and future research

We have studied the vertex-offset meshes of general polygonal meshes and show how their existence is dictated by face shape, surface geometry and surface topology. Our analysis is based on parallel spherical meshes, eigenvalue analysis of reflection transformations, and loop homotopy. We have provided an optimization method for computing the parallel spherical mesh of a given mesh, when it exists.

Further research problems include the following:

- The success of optimization methods for achieving face planarity depends strongly on initialization. It is well known that a circular quadrilateral mesh discretizes the principal curvature lines, so the initialization of a circular quad mesh should come from an appropriate sampling of the curvature lines. But for a general polygonal mesh, there is lack of research on how to go about this initialization.
- Due to aesthetic reasons, an open mesh from architecture design may have holes, which causes problems for computing vertex-offset meshes. One possible solution under investigation is to first fill the holes with EP meshes to make the mesh simply connected and then perform quasi-circular optimization and obtain the basis of parallel spherical meshes for computing discrete normal and offset structure. After that, the filled parts can be removed to restore the structure of the original the holes.

References

BOBENKO, A., AND SURIS, Y. 2005. Discrete differential geometry. consistency as integrability. http://arxiv.org/abs/math.DG/0504358.

LIU, Y., POTTMANN, H., WALLNER, J., YANG, Y.-L., AND WANG, W. 2006. Geometric modeling with conical meshes and developable surfaces. *ACM Transactions on Graphics(Proceedings of SIGGRAPH 2006) 25*, 3, 681 – 689.

MARTIN, R., DE PONT, J., AND T.J.SHARROCK. 1986. Cyclide surfaces in computer aided design. In *The Mathematics of Surfaces*, Clarendon Press, Oxford, J. Gregory, Ed., 253–268.

POTTMANN, H., AND WALLNER, J. 2008. The focal geometry of circular and conical meshes. *Adv. Comput. Math.*.

POTTMANN, H., LIU, Y., WALLNER, J., BOBENKO, A., AND WANG, W. 2007. Geometry of multi-layer freeform structures for architecture. *ACM Transactions on Graphics (Proceedings of SIGGRAPH 2007) 26*, 3, Article No. 65.